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ASYMPTOTIC ANALYSIS OF WAVE PROBLEMS FOR A CYLINDRICAL SHELL*

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The asymptotic solutions of elasticity theory equations for long and short-waves being propagated along the generator of a closed circular cylindrical shell, the normal waves, are examined. Two problems are investigated: Th Cauchy problem for an infinite shell and the problem of edge perturbation propagation in a semi-infinite shell. A general asymptotic solution is constructed for both problems. The asymptotic parameter is the shell wall thickness or the number of waves across this thickness. In the first problem, the frequency spectrum is determined, and in the second, the wave spectrum. Operators of the fundamental solutions are constructed for the problems considered.

On the whole, the solutions for shells are close to the solution of an analogous problem for a solid cylinder /1—3/, and especially to the results of /3/ in which quadratic bundles of operators generated by the problem were investigated. Below the asymptotic solution is constructed by analogy with /4/, but the order of the approximation is higher.

The results represented differ from the extensively utilized solutions based on the Timoshenko equations /5,6/. This refers expecially to the problem of edge perturbation propagation.

By using asymptotic representations, the quadratic bundle of operators analogous to /3/, are reduced to quadratic bundles of matrices whose spectral properties are investigated by using algebraic methods and perturbation theory /7,8/.

1. Formulation of the problem. Let us introduce the notation: R is the radius of the middle surface, $2h$ is the shell wall thickness, $\chi = h/R$, a and c are the velocities of volumetric and shear wave propagation, and $\gamma = c^2/a^2$, and $\gamma_1 = \gamma^{-1}$.

A cylindrical r, θ, z coordinate system is introduced, as is the corresponding displacement vector $q = (u, iv, w)^T$. Because of the cyclic periodicity the operator ∂_θ is replaced by the symbol $-in$, where n is an integer. This is actually a Fourier series in θ.

We construct the solution of the Cauchy problem in the form

$$q_\alpha(r, n, z, t) = \frac{(R^{1-\alpha}h^\alpha)^2}{2\pi} \int_{-\infty}^{\infty} \exp\left[i\frac{s(z, t, \lambda)}{R^{1-\alpha}h^\alpha}\right], \quad g_\alpha\left(\frac{r}{R^{1-\alpha}h^\alpha}, \lambda\right) d\lambda \tag{1.1}$$

Here $0 \leqslant \alpha \leqslant 1$ determines the degree of approximation, $s(z, t, \lambda) = \omega ct - \lambda z$ is the phase of the wave, where $\lambda R^{\alpha-1}h^{-\alpha}$ is the wave number, and $\omega(\lambda) cR^{\alpha-1}h^{-\alpha}$ is the frequency spectrum which we will determine.

The solution of the second problem on edge perturbation propagation in a semi-infinite shell $(z \geqslant 0)$ will be constructed in an analogous form

$$q_\alpha(r, n, z, t) = \frac{(R^{1-\alpha}h^\alpha)^2}{2\pi} \int^{\infty} \exp\left[i\frac{s(z, t, \omega)}{R^{1-\alpha}\alpha}\right], \quad g_\alpha\left(\frac{r}{R^{1-\alpha}h^\alpha}, \omega\right) d\omega \tag{1.2}$$

The expression for the wave phase $s(z, t, \omega)$ remains the same as for (1.1) but we will construct the wave spectrum $\lambda(\omega)$, $\omega \in (-\infty, \infty)$. We shall henceforth consider the spectra $\omega(\lambda)$ and $\lambda(\omega)$, exactly as the amplitude g_α, to depend everywhere on n as a parameter. The amplitudes $g_\alpha(\zeta, \omega)$ and $g_\alpha(\zeta, \lambda)$ themselves will be considered regular functions of ζ.

Let us introduce the dimensionless coordinate $\rho = rR^{\alpha-1}h^{-\alpha}$ and let the prime denote differentiation with respect to it. On the basis of (1.1) and (1.2) we can replace the operators ∂_t and ∂_z by the symbols $ic\omega R^{\alpha-1}h^{-\alpha}$ and $-i\lambda R^{\alpha-1}h^{-\alpha}$, after which the Lamé equations will be represented in the rather unusual form

$$Cg_\alpha'' + \left(\frac{D}{\rho} - i\lambda G\right)g_\alpha' + \left(\frac{F}{\rho^2} - \frac{i\lambda}{\rho}H - \lambda^2 S\right)g_\alpha + \omega^2 g_\alpha = 0, \quad C = \operatorname{diag}(\gamma_1, 1, 1), \quad S = \operatorname{diag}(1, 1, \gamma_1) \tag{1.3}$$

$$D = \begin{Vmatrix} \gamma_1 & n(\gamma_1 - 1) & 0 \\ n(\gamma_1 - 1) & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad G = \begin{Vmatrix} 0 & 0 & \gamma_1 - 1 \\ 0 & 0 & 0 \\ \gamma_1 - 1 & 0 & 0 \end{Vmatrix}$$

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$$F=\begin{Vmatrix}-\gamma_1-n^2 & n(\gamma_1+1) & 0\\ n(\gamma_1+1) & -1-n^2\gamma_1 & 0\\ 0 & 0 & -n^2\end{Vmatrix},\quad H=\begin{Vmatrix}0 & 0 & 0\\ 0 & 0 & n(\gamma_1-1)\\ \gamma_1-1 & n(\gamma_1-1) & 0\end{Vmatrix}$$

Let us introduce a vector equivalent to the stress vector in the plane $r=\text{const}$

$$\tau(r,\cdot)=\left(\frac{\sigma_{rr}}{a^2},\ \frac{i\sigma_{\theta r}}{c^2},\ \frac{\sigma_{rz}}{c^2}\right)^T=q_{,r}+\frac{A}{r}q+Bq_{,z} \tag{1.4}$$

Using the variable ρ in place of r and replacing ∂_z by its symbol, we obtain

$$\tau(\rho,\cdot)=R^{1-\alpha}h^{\alpha}\,[g_\alpha'+(A/\rho-i\lambda B)g_\alpha] \tag{1.5}$$

$$A=\begin{Vmatrix}1-2\gamma & -n(1-2\gamma) & 0\\ n & -1 & 0\\ 0 & 0 & 0\end{Vmatrix},\quad B=\begin{Vmatrix}0 & 0 & 1-2\gamma\\ 0 & 0 & 0\\ 1 & 0 & 0\end{Vmatrix}$$

No stresses on the free surfaces $r=R\pm h$ or $\rho=\chi^{-\alpha}\pm\chi^{1-\alpha}$ which are written as

$$g_\alpha'+(A/\rho-i\lambda B)g_\alpha=0,\quad \rho=\chi^{-\alpha}\pm\chi^{1-\alpha} \tag{1.6}$$

is the boundary condition for the ordinary differential equation (1.3).

In conjunction with the boundary conditions (1.6), equations (1.3) form a spectral problem for a quadratic bundle of operators. On the basis of results /3/ for a simply-connected domain, we consider the spectra $\omega(\lambda)$ and $\lambda(\omega)$ to have a single condensation point at infinity. In turn, this permits the construction of different approximations for the quadratic bundle of operators at regular points of the spectrum.

2. Long-wave approximation.

Let us take $\alpha=0$, and let us consider λ and ω to be sufficiently small in absolute value. Then (1.3) is defined in a narrow domain $\rho\in(1-\chi,\ 1+\chi)$ and we take the same approach as in /4/. To do this we expand $\tau(\rho,\cdot)$ in a Taylor series around the middle surface ($\rho=1$), while keeping up to four terms in the series. We combine conditions in the follows form:

$$\tau(1+\chi,\cdot)+\tau(1-\chi,\cdot)=0,\quad \tau(1+\chi,\cdot)-\tau(1-\chi,\cdot)=0$$

Now combining the expansions in the Taylor series, we obtain two equations

$$\tau(1,\cdot)+{}^1/_2\chi^2\tau''(1,\cdot)=O(\chi^4),\quad \tau'(1,\cdot)+{}^1/_6\chi^2\tau'''(1,\cdot)=O(\chi^4) \tag{2.1}$$

Let us introduce the notation $p_k=g_0^{(k)}(1,\cdot)$ and by using it and (1.5) we solve (2.1) for p_3 and p_4

$$p_3=-2\chi^{-2}(p_1+Mp_0)+O(1),\quad p_4=-6\chi^{-2}(p_2+Mp_1)-Ap_3+O(1),\quad M=A-i\lambda B \tag{2.2}$$

We differentiate (1.3) once and then twice with respect to ρ, referring the result to $\rho=1$

$$Cp_3+\omega^2p_1+\ldots=0,\quad Cp_4+\omega^2p_2+\ldots=0 \tag{2.3}$$

Components with the lowest terms are omitted in these equations. From (2.2) and (2.3) we obtain equations in p_1 and p_2 (E is the unit 3×3 matrix)

$$(E-{}^1/_6\omega^2\chi^2C^{-1})p_2=-Mp_1+Ap_0+O(\chi^2),\quad (E-{}^1/_2\omega^2\chi^2C^{-1})p_1=-Mp_0+O(\chi^2) \tag{2.4}$$

Let $|\omega^2\chi^2|<1$, which is in complete agreement with the assumption about the smallness of ω and even weakens it substantially. For instance, $|\omega^2\chi^2|$ can be considered a small number and then (2.4) can be solved as follows:

$$p_1=-(E+{}^1/_2\omega^2\chi^2C^{-1})Mp_0+O(\chi^2+\omega^4\chi^4),\quad p_2=(E+{}^1/_6\,\omega^2\chi^2C^{-1})(Ap_0-Mp_1)+O(\chi^2+\omega^4\chi^4) \tag{2.5}$$

There now remains to substitute the expressions obtained into (1.3), referring it to $\rho=1$. But first, p_1 in the second expression in (2.5) should certainly be replaced by p_0 by using the first expression. We finally obtain the system of linear algebraic equations

$$[T_0(\lambda)+\omega^2(E+\chi^2T_{1(\omega)}(\lambda))]p_0=O(\chi^2+\omega^4\chi^4) \tag{2.6}$$

Here $T_j(\lambda)$ $(j=0,1)$ are quadratic bundles of matrices

$$T_j(\lambda)=P_j-i\lambda K_j-\lambda^2L_j,\quad j=0,1 \tag{2.7}$$

We present just the structure of the bundle $T_0(\lambda)$

$$P_0=\begin{Vmatrix}-4(1-\gamma) & 4n(1-\gamma) & 0\\ 4n(1-\gamma) & -4n^2(1-\gamma) & 0\\ 0 & 0 & -n^2\end{Vmatrix},\quad K_0=\begin{Vmatrix}0 & 0 & -2(1-2\gamma)\\ 0 & 0 & n(3-4\gamma)\\ 2(1-2\gamma) & -n(3-4\gamma) & 0\end{Vmatrix},\quad L_0=\operatorname{diag}(0,\ 1,\ 4(1-\gamma))$$

The bundle $T_{1(4)}$ is formed as follows:

$$T_{1(4)}(\lambda) = {}^1/_6\,[2\,(CM - N)C^{-1}M + M^2 + A], \quad N = D - i\lambda G$$

The matrix coefficients of the bundles T_0 and $T_{1(4)}$ are real matrices. The subscript (4) indicates the order of approximation of the boundary conditions (1.6). The fact is that the order of the approximation $O(\chi^3)$ will result in an analogous result, in which $T_{1(3)}(\lambda)$ will replace $T_{1(4)}(\lambda)$. For this it is sufficient to take $\tau''' = 0$ in (2.1). An approximation $O(\chi^2)$ is used in /4/ and a simpler axisymmetric problem is solved. For such an approximation $T_{1(2)}$ will be the zero-th matrix. The matrix $T_{1(3)}$ is constructed thus:

$$T_{1(3)}(\lambda) = {}^1/_2\,(CM - N)C^{-1}M$$

The bundle $T_0(\lambda)$ is comprised of the symmetric matrix P_0, the skew-symmetric matrix K_0 and the diagonal degenerate matrix L_0.

Therefore, for real λ the bundle T_0 is Hermitian, negative-definite, and λ is singular. The bundles $T_{1(3)}$ and $T_{1(4)}$ do not possess the property of being Hermitian. Because χ^2 is a small number, perturbation theory can be used, but it would be desirable to reduce the bundle (2.6) to an analytic perturbation of the bundle $T_{0\omega}(\lambda) = T_0(\lambda) + \omega^2 E$. This can be done by linear interpolation by introducing the bundle

$$T_1(\lambda) = (k_1 + k_2)^{-1}\,[k_1 T_{1(3)}(\lambda) + k_2 T_{1(4)}(\lambda)]$$

Let us require that such a bundle be Hermitian. This can always be done in the general case. For $\gamma = {}^1/_3$ (the Poisson's ratio equals 0.25) the condition that $T_1(\lambda)$ be Hermitian will be $k_1 = -{}^5/_9$ and $k_2 = 1$. In fact, there is an extrapolation here towards reducing the order of the approximation of the boundary conditions (1.6), which became $3k_1 + 4k_2 = {}^7/_3$. But here the general order of approximation of the operator, $O(\chi^2)$, was conserved.

Finally, discarding small terms in the right side in (2.6) and introducing the bundle $T_1(\lambda)$, we arrive at the spectral problem for the analytically perturbed quadratic bundle of matrices

$$T_\omega(\lambda) = T_0(\lambda) + \omega^2\,[E + \chi^2 T_1(\lambda)] \tag{2.8}$$

For $\gamma = {}^1/_3$ the bundle T_1 is represented as follows:

$$T_1(\lambda) = \frac{1}{72}\begin{Vmatrix} 2 + 36n^2 + 36\lambda^2 & -10n & -i\lambda \\ -10n & 18 - 8n^2 & 2i\lambda n \\ i\lambda & -2i\lambda n & -8\lambda^2 \end{Vmatrix}$$

3. The Cauchy problem.

Let us examine this problem in the long-wave band when the initial data

$$q_0(R, n, z, 0) = \varphi_{0n}(z), \quad q_0^{\cdot}(R, n, z, 0) = \varphi_{1n}(z)$$

are given. In turn, this means that the initial data are given in the form of an inverse Fourier transform in the coordinate z: $p_0(\lambda) = \Phi_0(-\lambda)$; $\omega(\lambda)p_0(\lambda) = \Phi_1(-\lambda)$, where $\Phi_0(\lambda)$ and $\Phi_1(\lambda)$ are the Fourier transforms for φ_{0n} and φ_{1n}. For this reason the frequency spectrum $\lambda(\omega)$, $\lambda \in (-\infty, \infty)$ should be constructed.

In addition to the bundle $T_\omega(\lambda)$ in (2.8), we consider the bundle $T_{0\omega}(\lambda) = T_0(\lambda) + \omega^2 E$. For any real λ the bundle $T_0(\lambda)$ is Hermitian and negative-definite. Therefore, there are three eigenvalues $\omega_{0j}^2(\lambda) > 0$ to which the unitary matrix of eigenvectors $Q(\lambda)$ corresponds. The bundle $T_\omega(\lambda)$ can be considered as an analytic perturbation of the bundle $T_{0\omega}(\lambda)$ /7/. Then the estimate

$$\omega_j^2(\lambda) = \omega_{0j}^2(\lambda)\,[1 - \chi^2 e_j^T T_1(\lambda) e_j] + O(\omega_{0j}^4 \chi^4) \tag{3.1}$$

holds for the eigenvalues of this bundle $\omega_j^2(\lambda)$, where $e_j = e_j(\lambda)$ is the eigenvector of $T_{0\omega}(\lambda)$ or the j-th column of the matrix $Q(\lambda)$.

In the limits $O(\omega^4\chi^4)$ the eigenvectors of both bundles agree. We select the positive values of $\omega_j(\lambda)$, $j = 1, 2, 3$ and we form the diagnonal matrix

$$\Omega(\lambda) = \operatorname{diag}\,[\omega_1(\lambda),\ \omega_2(\lambda),\ \omega_3(\lambda)] \tag{3.2}$$

Now the operator (1.1) is constructed in the form of the diverging waves

$$q_0(R, n, z, t) = \frac{R}{2\pi}\int_{-\infty}^{\infty} Q(\lambda)\left\{\exp\left[i\,\frac{ct\Omega(\lambda) - \lambda z E}{R}\right]\varphi^+(\lambda) + \exp\left[-i\,\frac{ct\Omega(\lambda) + \lambda z E}{R}\right]\varphi^-(\lambda)\right\}d\lambda \tag{3.3}$$

From initial data in the form Φ_0 and Φ_1 we determine φ^+ and φ^- and finally the general solution is constructed as a convolution

$$q_0(R, n, z, t) = D_n^{\cdot}(z, t) * \varphi_{0n}(z) + D_n(z, t) * \varphi_{1n}(z) \tag{3.4}$$

$$D_n(z, t) = \frac{1}{4\pi c}\int_{-\infty}^{\infty} \exp\left(\frac{i\lambda z}{R}\right) Q(\lambda) \sin\left(\frac{ct}{R}\,\Omega(\lambda)\right) \times \Omega^{-1}(\lambda)\, Q^*(\lambda)\, d\lambda$$

The dependence on n in the last expression under the integral should be understood as on a parameter.

4. Problem of edge perturbations. We solve this problem, as the preceding one has been, in the low-frequency band of the perturbations applied to the edge $z = 0$ in the form $q_0(R, n, 0, t) = f_n(t)$ with the spectral function $F_n(\omega)$, $\omega \in (-\infty, \infty)$. To construct the solution, we determine the wave spectrum $\lambda(\omega)$ for which we consider the bundle $T_\omega(\lambda)$ in the following form:

$$T_\omega(\lambda) = P - i\lambda K - \lambda^2 L, \quad L = L_0 + \omega^2\chi^2 L_1, \quad P = P_0 + \omega^2 E + \omega^2\chi^2 P_1, \quad K = K_0 + \omega^2\chi^2 K_1 \qquad (4.1)$$

The characteristic equation is the bicubic equation $|T_\omega(\lambda)| = 0$, that has six roots with possible multiplicity taken into account, that are disposed symmetrically relative to the real and imaginary axes of the complex plane. Only three roots $\lambda_j(\omega)$, $j = 1, 2, 3$, located in the positive part of the real axis and opening into the lower complex half-plane, satisfy the radiation conditions.

Let a normal Jordan form of the 3×3 matrix $J(\omega)$ correspond to these roots. This matrix is constructed by means of elementary divisors of the bundle $T_\omega(\lambda)$. Let us form the 6×6 matrix $J_6 = \operatorname{diag}(J(\omega), -J(\omega))$, and let us construct an equivalent transformation of the bundle $T_\omega(\lambda)$. The equivalent transformation of the linear bundle $J_6 - \lambda E_6$ corresponds to it, where E_6 is the 6×6 unit matrix.

Let us introduce the auxiliary vector x in three-dimensional complex space, and let us convert the spectral problem for T_ω in the form (4.1) into a spectral problem for the linear bundle of 6×6 matrices

$$\left[\begin{Vmatrix} 0 & E \\ P & -K \end{Vmatrix} - i\lambda \begin{Vmatrix} E & \\ & -L \end{Vmatrix}\right] \begin{Vmatrix} p_0 \\ x \end{Vmatrix} = 0 \qquad (4.2)$$

For $|L| \neq 0$ this problem is equivalent to the problem about the spectrum of the matrix

$$\Gamma_\omega = \begin{Vmatrix} 0 & E \\ -L^{-1}P & L^{-1}K \end{Vmatrix}$$

Here $|\Gamma_\omega - \lambda E_6| = -|L|^{-1}|T_\omega(\lambda)|$. It hence follows that there exists a nondegenerate transformation U independent of λ such that $UJ_6U^{-1} = \Gamma_\omega$ /8/.

Let $X(\omega)$ denote a 3×3 matrix of eigen- and associated vectors for $J(\omega)$. Then $X_6 = \operatorname{diag}(X(\omega), -X(\omega))$ will be the matrix of eigen- and associated vectors for J_6, and $Y_6 = UX_6$ will be the same matrix for Γ_ω.

Only the upper half of the matrix Y_6, the three upper rows, will correspond to the vector p_0 while the rest will correspond to the vector x. Only the first three columns of Y_6 will correspond to the eigenvalues satisfying the radiation conditions. Consequently, only the left upper 3×3 block of the matrix Y_6, to be denoted by $Y(\omega)$, should be considered eigenvectors of the bundle T_ω. This matrix is represented in the form $Y(\omega) = Q(\omega)X(\omega)$, where $Q(\omega)$ is a 3×3 matrix satisfying the equation

$$PQ - iKQJ(\omega) - LQJ^2(\omega) = 0 \qquad (4.3)$$

The nondegenerate solution of this equation exists to the accuracy of the similarity transformation /8/. This latter circumstance turns out to be negligible for the subsequent construction.

Now (1.2) can be written thus:

$$q^0(R, n, z, t) = \frac{R}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\exp\left[i\,\frac{\omega ctE - zJ(\omega)}{R}\right]\psi_n(\omega)\,d\omega$$

The function ψ_n is determined from the boundary condition for $z = 0$, and the final result is written in the form of a convolution in t:

$$q_0(R, n, z, t) = D_n^*(z, t)_0 f_n(t), \quad D_n^*(z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(i\,\frac{\omega ct}{R}\right)Y(\omega)\exp\left[-i\,\frac{zJ(\omega)}{R}\right]Y^{-1}(\omega)\,d\omega \qquad (4.4)$$

The subscript n indicates dependence of the integrand on n as on a parameter.

An investigation of the bundle $T_\omega(\lambda)$ as an operator in which λ is replaced by $iR\partial_z$ is of special value. In this case the equation for $T_\omega(\lambda)p_0$ is a singular perturbation of the equation $[T_0(i\partial_z) + \omega^2 E]p_0 = 0$. The fact is that the characteristic equation $|T_0(\lambda) + \omega^2 E| = 0$ is a biquadratic with the roots $\pm\lambda_{01}(\omega)$, $\pm\lambda_{02}(\omega)$. We consider the roots with the plus sign satisfy the radiation conditions. The corresponding roots of the bundle $T_\omega(\lambda)$ are a regular perturbation of the roots λ_{01} and λ_{02} and satisfy the estimate

$$|\lambda_j^2(\omega) - \lambda_{0j}^2(\omega)| = O(\omega^2\chi^2), \quad j = 1, 2$$

The third root in the characteristic equation, to be denoted by $\pm\lambda_3(\omega)$ is a consequence of the singularity, and is written thus in general form:

$$\lambda_3^2(\omega) = 2(1-\gamma)\chi^{-2}(\omega^2 - n^2)[\omega^2 - 4(1-\gamma)][\omega^2 - 4(1-\gamma)(n^2+1)] + O(1)$$

Therefore, for $n \geqslant 1$ and small ω this root has a large imaginary part, which means the formation of a rapidly damped wave in a narrow boundary layer zone around $z = 0$. However, for relatively large ω (on the order of n), the formation of slow undamped waves is possible.

Still another singularity appears for multiple $\lambda_1 = \lambda_2$. Then a component containing z appears in the matrix exponential, and the phenomenon of quasiresonance will be observed in the wave pattern. This is characterized by the fact that the amplitude maximum will be at a certain distance from the site of application of the perturbation, and will subsequently drop exponentially.

5. Short-wave asymptotic. In the short-wave band λ and ω are sufficiently large. Hence, we take $\alpha = 1$ to construct the solution in the form (1.1) and (1.2), but the boundary conditions (1.6) will here be given for $\rho = \chi^{-1} \pm 1$. In this case the Taylor series expansion does not yield a satisfactory result for a small number of terms in the series. Hence, we go over to the new argument $\rho' = \rho - \chi^{-1}$. We represent (1.3) and the boundary conditions (1.6) as follows:

$$Cg_1'' - i\lambda Gg_1' - \lambda^2 Sg_1 + \omega^2 g_1 = O(\chi^2), \quad g_1' - i\lambda Bg_1 = O(\chi), \quad \rho' = \pm 1 \tag{5.1}$$

In such a formulation, the boundary conditions which have an error $O(\chi)$ assure compliance with conditions (1.6) with an error $O(\chi^2)$. Discarding the small terms in the right side of the operator (5.1), we arrive at the spectral problem for the operator describing wave propagation in an unbounded plate of thickness $2h$. The solution of such a boundary value problem for an ordinary differential equation will be constructed in the form $g_1 = \exp(\mu\rho')$. This results in such a characteristic equation

$$|\mu^2 C - i\lambda\mu G - \lambda^2 S + \omega^2 E| = 0$$

Here there are six roots in all

$$\mu_{1,4}^2 = \lambda^2 - \gamma\omega^2, \quad \mu_{2,5}^2 = \mu_{3,6}^2 = \lambda^2 - \omega^2$$

We construct the general solution in the form

$$g_1 = \exp\left(\frac{\zeta V}{h}\right)\varphi_+ + \exp\left(-\frac{\zeta V}{h}\right)\varphi_-, \quad \zeta = r - R, \quad V = \operatorname{diag}(\mu_1, \mu_2, \mu_3) \tag{5.2}$$

Hence substituting (5.2) into the boundary conditions results in a homogeneous system of equations

$$Ze^V\varphi_+ - \bar{Z}e^{-V}\varphi_- = 0, \quad Ze^{-V}\varphi_+ - \bar{Z}e^V\varphi_- = 0, \quad Z = V - i\lambda B, \quad \bar{Z} = V + i\lambda B$$

One of the solutions of the characteristic equation of such a system is $\mu_2 = 0$ or $\lambda_2 = \omega$. This corresponds to a shear wave in the plane $r = \text{const}$ being propagated at its natural velocity c. Two other solutions are obtained for $\omega \to \infty$ from the equation $|Z| = |\bar{Z}| = 0$. This is the equation for Rayleigh waves. However, we obtain the equation in the form

$$|e^U\bar{Z}_1^{-1}Z_1e^U - e^{-U}\bar{Z}_1^{-1}Z_1e^{-U}| = 0, \quad U = \operatorname{diag}(\sqrt{\lambda^2 - \gamma\omega^2}, \sqrt{\lambda^2 - \omega^2})$$

$$Z_1 = U - i\lambda B_1, \quad \bar{Z}_1 = U + i\lambda B_1, \quad |Z_1| = |\bar{Z}_1| \neq 0, \quad B_1 = \begin{Vmatrix} 0 & 1-2\gamma \\ 1 & 0 \end{Vmatrix}$$

for not extremely high frequencies. This transcendental equation always has one real positive solution $\lambda_1^2(\omega) > \gamma\omega^2$. As regards the other solution $\lambda_3(\omega)$, it can be a complex-valued function. The latter constructions do not differ essentially from those presented Sects. 3 and 4.

6. Concluding remarks. To a known error, asymptotic solutions permit the construction of the solution for any initial data and edge perturbations. A particular case is the axisymmetric problem when $n = 0$, considered earlier in /4/. The general operator for $n = 0$ is split into two in the solutions constructed. One describes normal waves, and the other torsion waves. The results agree qualitatively with the solution for a continuous cylinder /3/.

Let us examine the agreement with an analogous solution on the basis of the Timoshenko equations /5,6/. We limit ourselves only to the axisymmetric case ($n = 0$) and we present two dispersion equations, obtained above and represented in /6/. We omit the regular perturbation of the coefficients by retaining just the singular perturbation for a higher degree of λ ($\gamma = {}^1/_3$)

$$(\omega^2 - \lambda^2)[\lambda^4 {}^3/_2\omega^2\chi^2 - \lambda^2 4(5 - 2\omega^2) + \omega^2(3\omega^2 - 8)] = 0 \tag{6.1}$$

An analogous equation borrowed from /6/ and reduced to the notation used above is:

$$(\omega^2 - \lambda^2)[\lambda^6 64\chi^2 / 9 - \lambda^4 8\omega^2\chi^2 / 3 - \lambda^2 4 (5 - 2\omega^2) + \omega^2 (3\omega^2 - 8)] = 0 \qquad (6.2)$$

In both cases the first factor corresponds to the torsion problem, and the characteristic equation for the normal waves is contained in the brackets.

For the Cauchy problem, the frequency spectrum $\omega(\lambda)$ agrees in practice in both equations, the difference is on the order of $O(\chi^2)$. But the error in (6.1) is $O(\chi^2 + \omega^2\chi^4)$ with respect to the actual spectrum.

There are substantial differences for the wave spectrum of the edge perturbation problem. There are two roots in (6.1), $\lambda_1(\omega)$ and $\lambda_3(\omega)$, where $\lambda_1(\omega)$ is the regular root and $\lambda_3(\omega)$ is the singular root. Let us recall that the root $\lambda_2 = \omega$ correspond to torsion and is identical for (6.1) and (6.2). Hence, the problem considered above for the edge perturbation is solvable for any conditions at the end $z = 0$.

In addition to $\lambda_2 = \omega$ in (6.2) there is the regular solution $\lambda_1^2 = {}^1/_4\omega^2 \times (3\omega^2 - 8) / (2\omega^2 - 5)$, which agrees with the analogous solution for (6.1). There is still another pseudoregular solution $\lambda_3^2 = {}^3/_8\omega^2 + O(\chi^2)$. But there is still another singular solution also $\lambda_4^2 = \omega^2 / (8\chi^3) + O(1)$, which agrees completely with the eighth order equation (6.2). Therefore, the wave spectrum consists of four branches, and the frequency spectrum of three, as assumed. The essential qualitative distinction here is that which cannot assure a solution in the edge perturbation problem for any edge perturbations.

In conclusion, we note that the proposed asymptotic analysis is applicable for any degrees of approximation α. Thus for $\alpha = {}^1/_2$ we arrive at asymptotic solutions relative to $\sqrt{Rh}$.

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